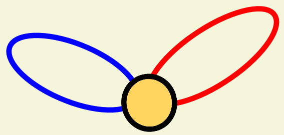
General question:

What is the smallest number of vertices in a homomorphic image satisfying property P?

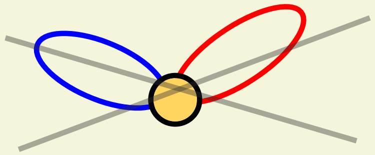
P: no condition,

then always 1.

$K_1^{\pm}$ 

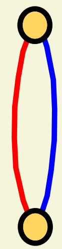
General question:

What is the smallest number of vertices in a homomorphic image satisfying property P?

P: no $K_1^{\pm}$, 

then we have a difficult parameter to study.

In this case if we replace every edge of a graph by a digon, then we get the notion of the chromatic number of graphs.



Definition.

Given a signed graph (G, σ) with no $K_1^{\pm}$, we define its complex chromatic number, denoted $\chi_{com}(G, \sigma)$, to be the minimum number of vertices of a homomorphic image with no $K_1^{\pm}$.

Examples

$\chi_{com}(K_3, \sigma) = 1$.

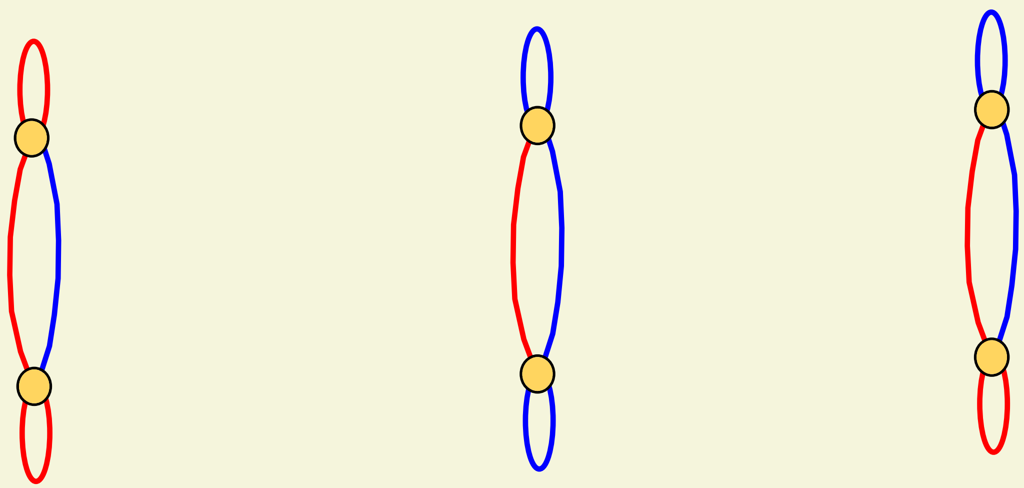
$\chi_{com}(C_{-4}) = 2$.

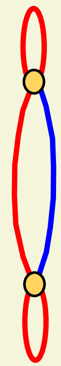
Question.

What is the smallest value of n such that $\chi_{com}(K_n, \sigma) \geq k$ for some signature?

Theorem. There exists a signed planar simple graph whose complex chromatic number is 3.

That is to say there exists a planar signed graph (G, σ) which does not map to any of:

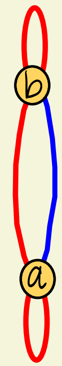


Construction of a signed planar simple graph which does not map to: 

Note: It was conjectured by Macajova, Raspaud & Skoviera, in an equivalent formulation, that every signed planar simple graph maps to this target.

first construction was given by Kardos & Narboni.

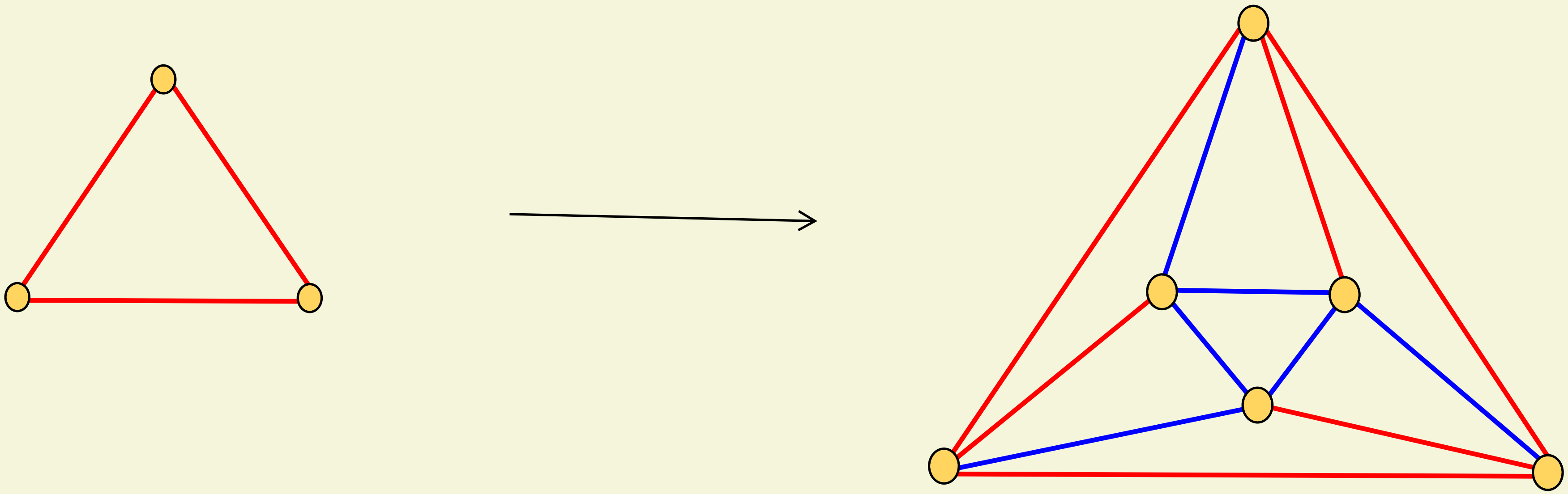
Construction of a signed planar simple graph which does not map to:

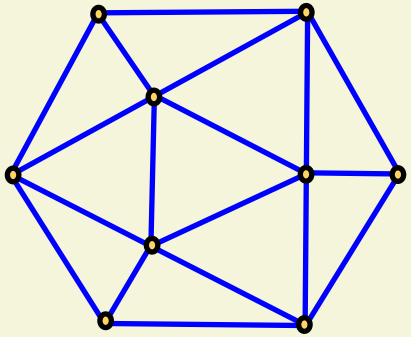


Triangle property.

We can always ensure that:

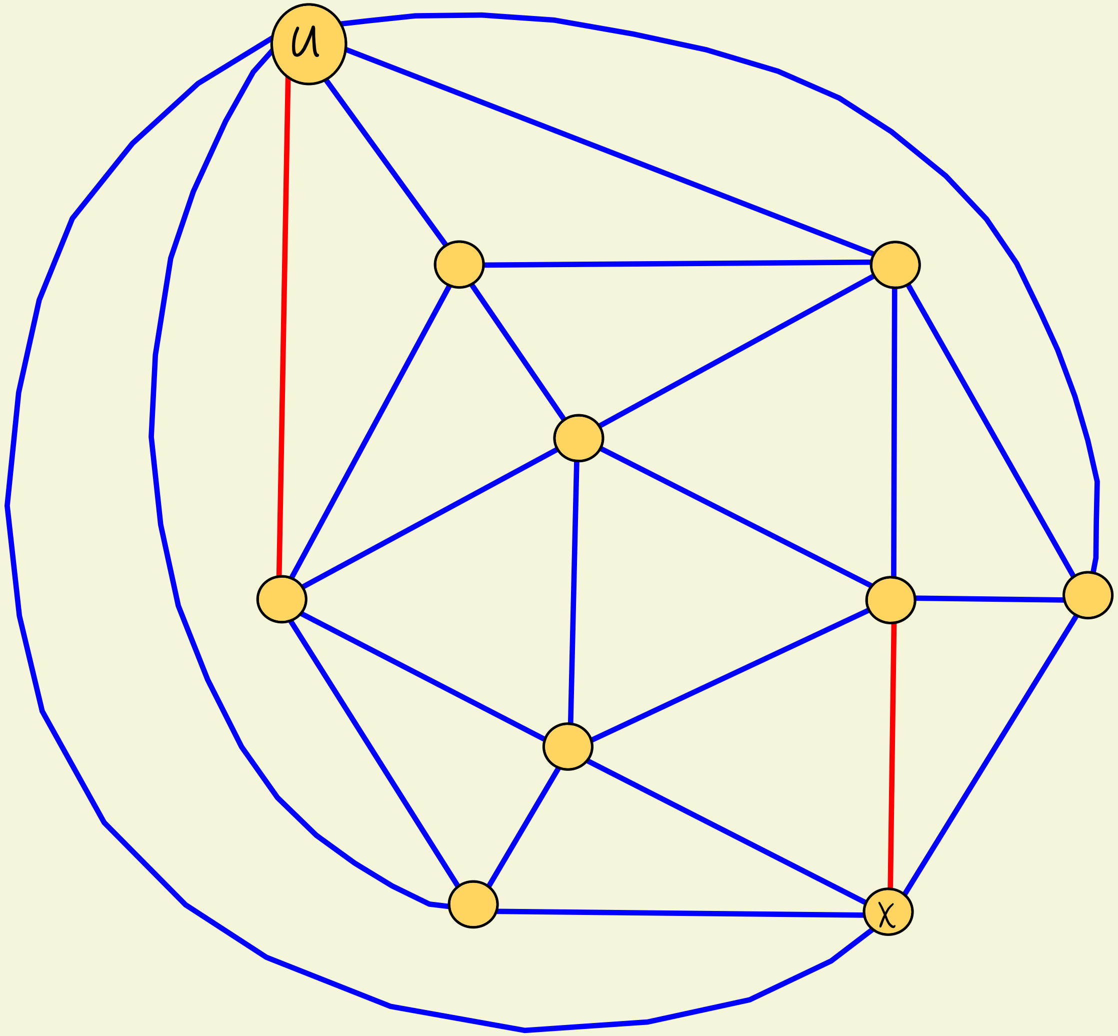
Vertices of a facial cycle are not all mapped to a same vertex.

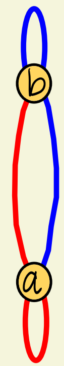




Gadget:

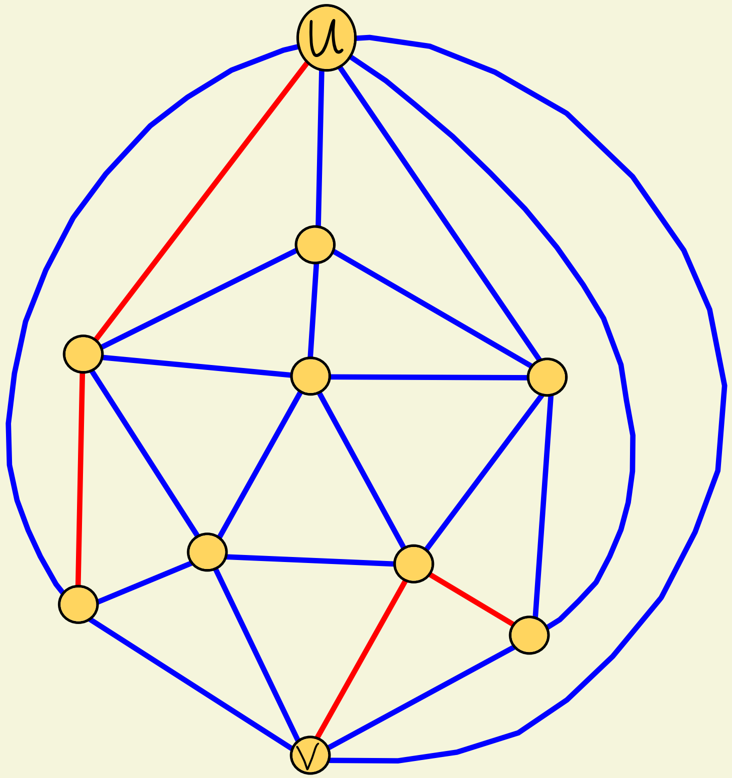
Any mapping of the gadget must map u & x to different vertices.

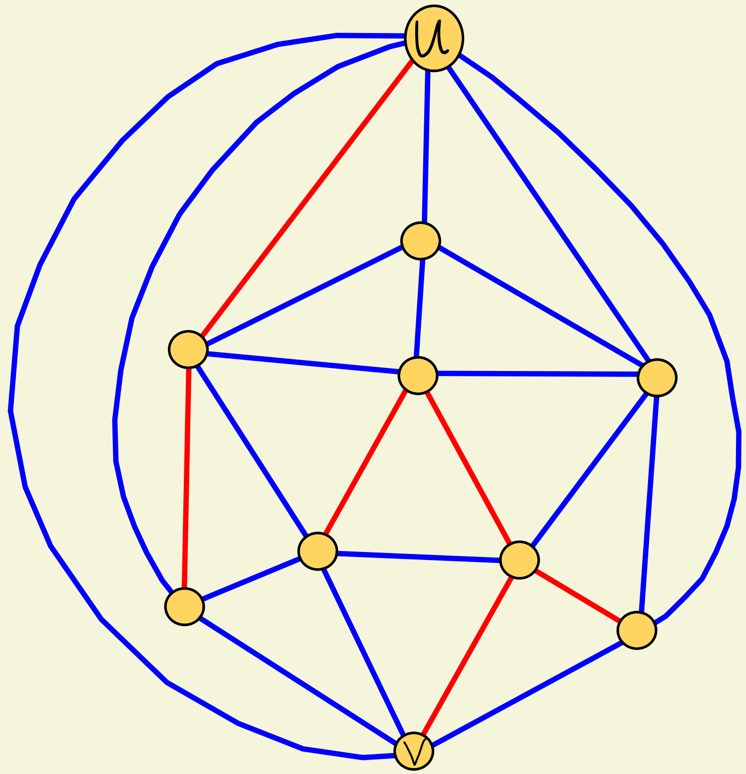


Construction of a signed planar simple graph which does not map to: 

Note: that every signed planar (simple) graph maps to this target was asked/conjectured by Jiang & Zhu in an equivalent formulation.

Remark. Triangle property can be ensured similarly.





If a 2-coloring satisfies triangle property and no negative even cycle is monochromatic, then there exists a monochromatic positive 5-cycle containing the edge uv.

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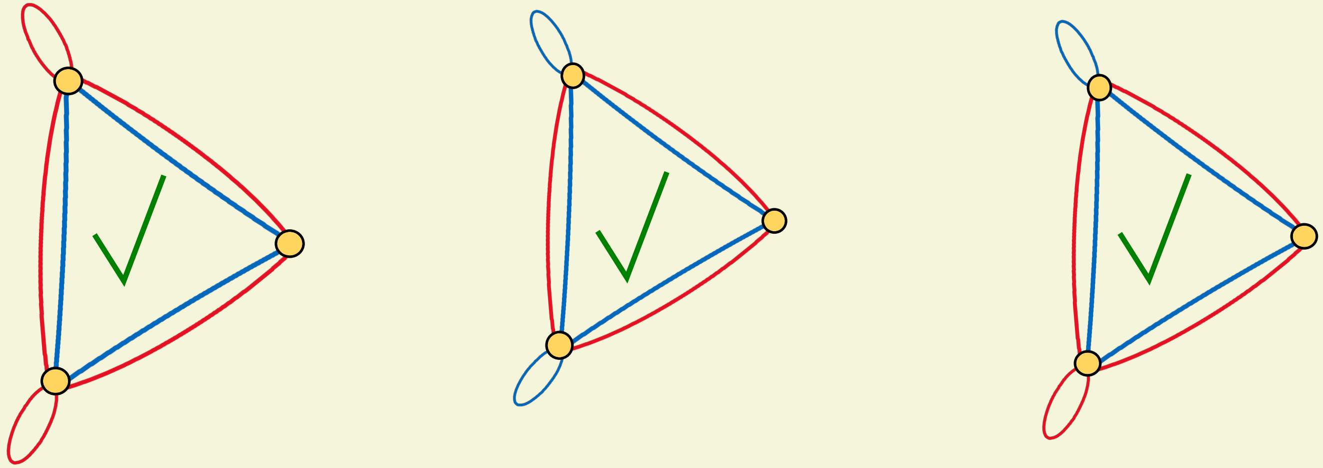
To complet the construction, we merge the two by identifying "u", "v" and "uv". The result is a signed planar simple graph $\hat{G}$ with a special pair u & v and of vertices with the following property:

In any 2-coloring of $\hat{G}$ where u and are colored by a same color, and triangle property is satisfied, there exists a monochromatic negative even cycle.

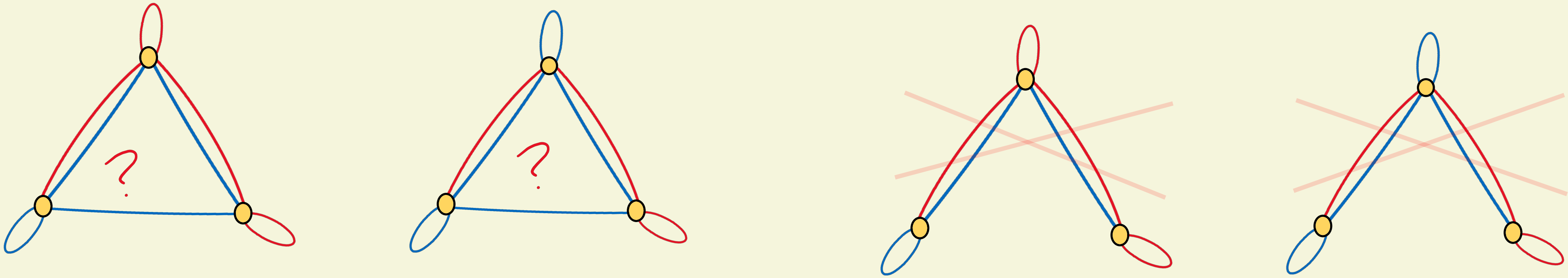
A final construction is obtained by replacing each edge of triangle with a copy of this gadget.

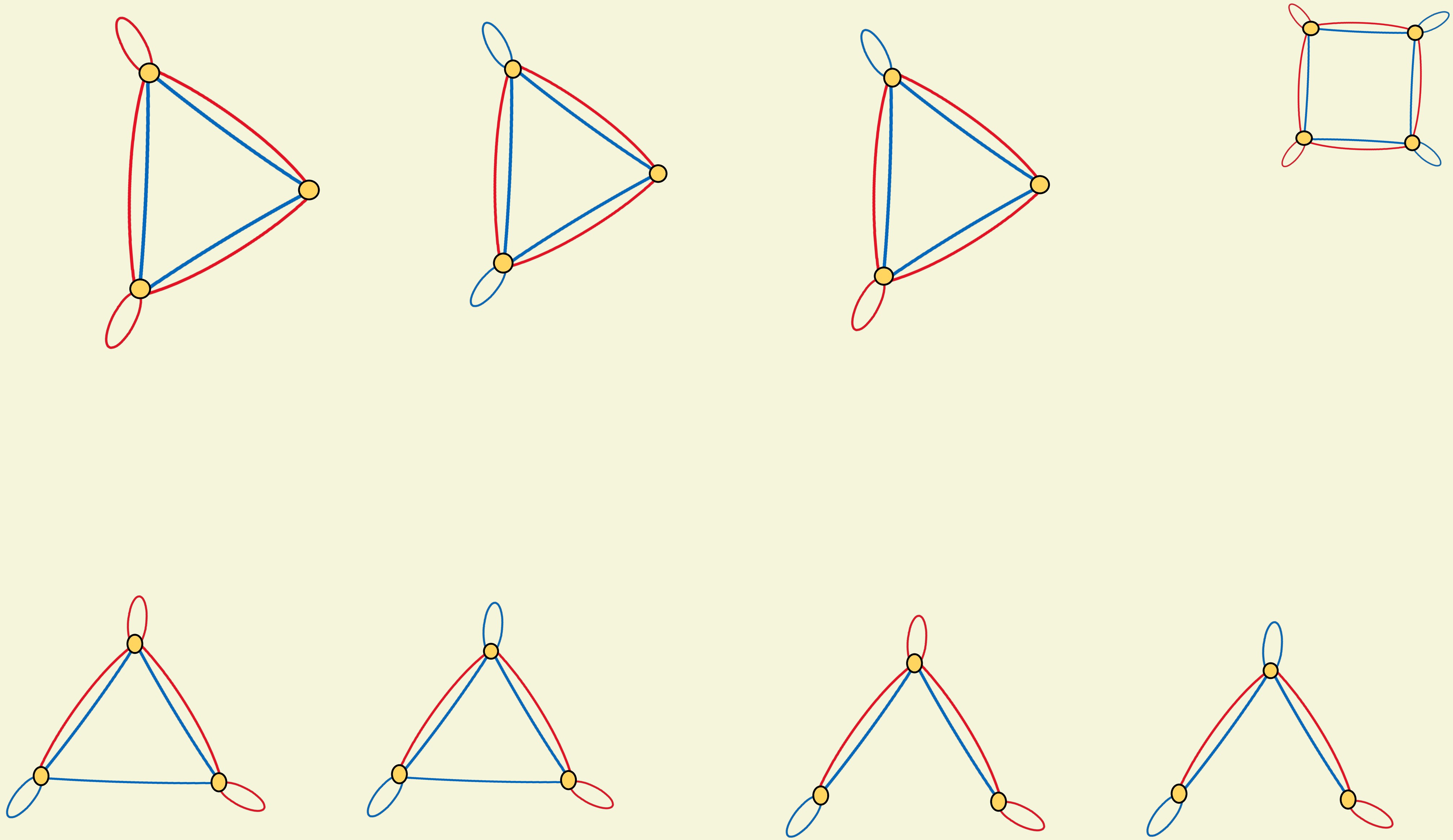
Corollary. There exists no $K_1^{\pm}$-free signed graph on 2 vertices to which every signed planar simple graph admits a homomorphism.

Question. What about $K_1^{\pm}$-free signed graphs on 3 vertices?



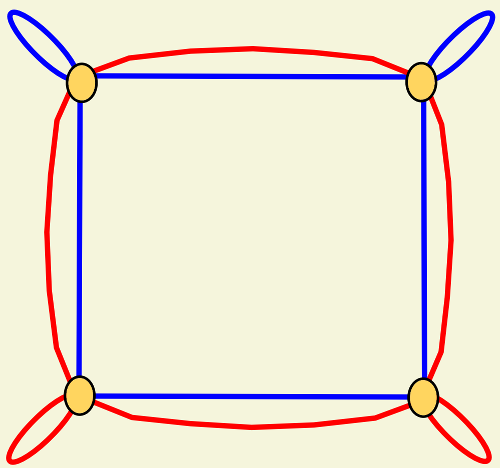
And, moreover, they are minimal i.e. no subgraph works.

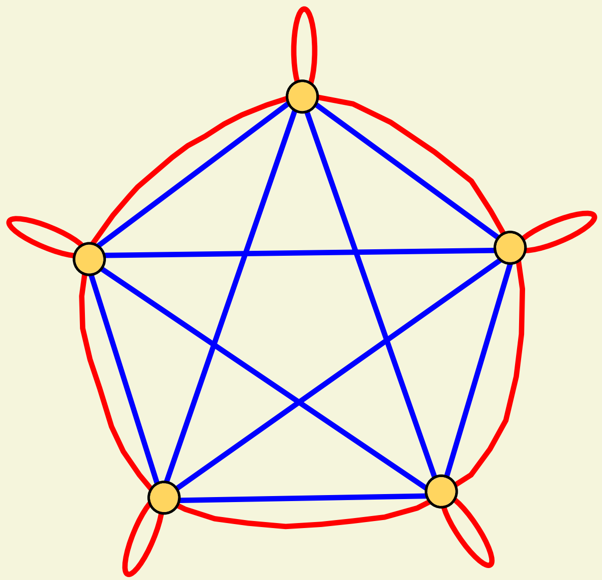


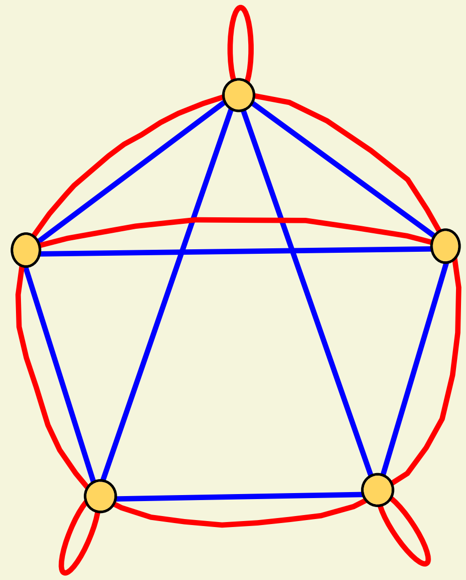


Bounding signed planar simple graphs:

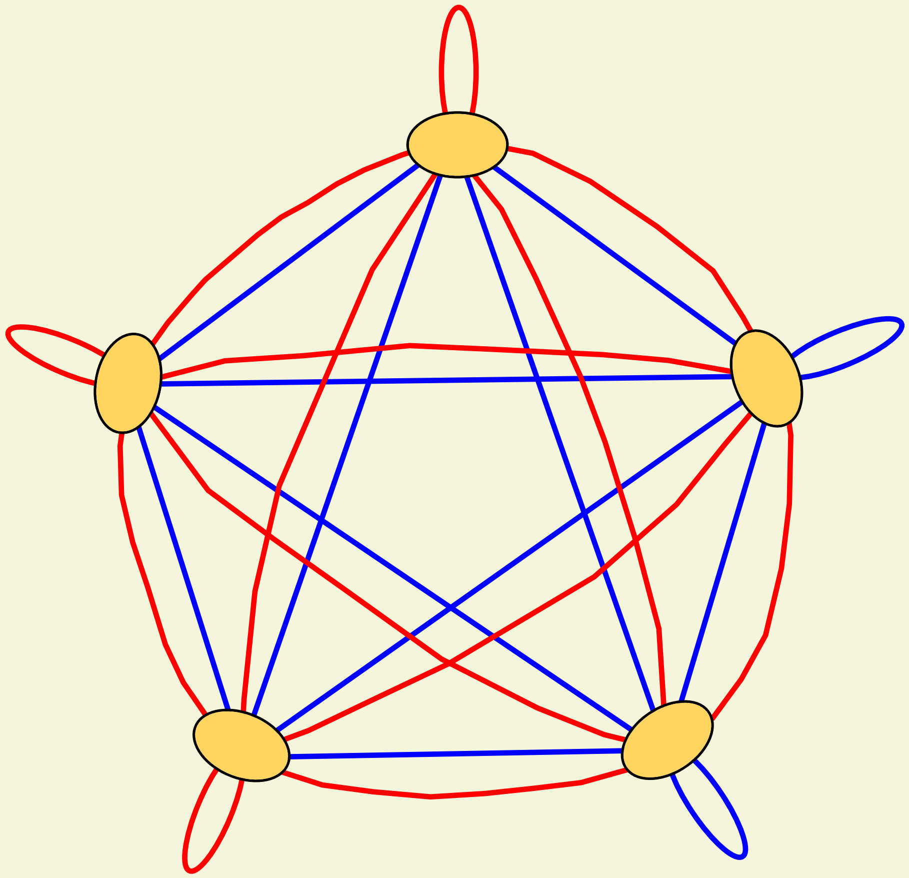
Other targets of high interest (allowing digon and loops).







Why "complex chromatic number"?



Definition. Let $SR(s,t)$ be the smallest value of n satisfying:

For any signature σ on K_n, the signed graph (K_n, σ) contains either a balanced K_s or an antibalanced K_t.

Chromatic number for  -free signed graph:

smallest number of vertices in a -free homomorphic image × 2.

Chromatic number for  -free signed graph:

smallest number of vertices in a -free homomorphic image x 2.

First studied by Zaslavsky in 1980's.

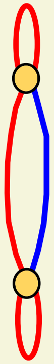
(in equivalent form and as a natural extension of proper coloring)

Recent developments following the paper of Macajova, Raspand & Skoveira.

Relation to list coloring

Theorem. If for any signature σ, the signed graph

[Zhu, 2020]

(G, σ) is  -colorable, then for any 2-list assignment L on G, there exist an L-coloring of G where each color class is a bipartite graph.

Conjecture. Given a planar graph and a 2-list assignment

[Kundgen, Ramamurthi, 2002]

there exists such a list coloring

Chromatic number for loop-free signed graph

both  and  are forbidden

the chromatic number of the underlying graph

Double Switch Graph $DSG(G, 6)$

Double Switch Graph $DSG(G,\sigma)$

Theorem. $(G,\sigma) \longrightarrow (H,\pi)$ if and only if $(G,\sigma) \xrightarrow{\text{s.p.hom}} DSG(H,\pi)$

Moreover (H,π) is a switch-core if and only if $DSG(H,\pi)$ is a S.P-core.

[Brewster & Graves]

Alon-Marshal signed graph AM(k)

Vertices: k-tuples $\overbrace{(i, +/-, +/-, \ldots, *, \ldots, +/-)}^{k}$

$\{1, 2, 3, \ldots, i, \ldots, k\}$

edges: $(i, +/-, +/-, \ldots, \overset{i}{*}, \ldots, \overset{j}{+/-}\ \ldots, +/-) \sim (j, +/-, +/-, \ldots, \overset{i}{+/-}, \ldots, \overset{j}{*} \ldots, +/-)$

if $i \neq j$ and the sign is the product of $\times$.

Note: The construction is analogue of Raspaud-Sopena oriented graph RS(k)

Theorem. If G is a graph of acyclic chromatic number $\leq k$,

[Alon & Marshall]

then (G, σ) admits an edge-sign preserving mapping to $AM(k)$ for any choice of σ.

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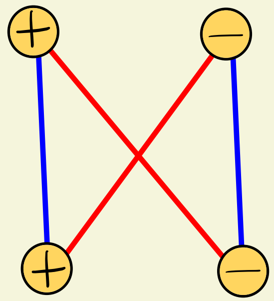
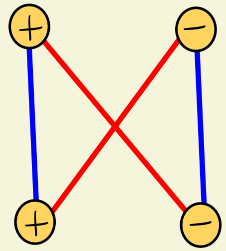
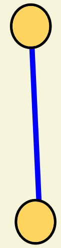
[Alon & Marshall] then (G, σ) admits an edge-sign preserving mapping to $AM(k)$ for any choice of σ.

Lemma. Every signed tree admits an edge sign-preserving mapping to



Observation.

 $= DSG\Big($  $\Big)$

Strengthening of the previous theorem. [proposed by Zaslavsky (in the class)]

If vertices of (G, σ) are k-colored such that:

- each color class is an independent set
- each pair of colors induce a balanced subgraph,

then (G, σ) admits an edge-sign preserving homomorphism to $AM(k)$.

Note. K_6-M is an example of a planar graph whose acyclic chromatic number is 5.

One can assign a signature to this graph such that it would also need 5-colors to satisfy the above condition.

Question. Would 4 colors be enough for every signed planar triangle-free simple graph?

Theorem. Every planar graph is acyclically 5-colorable.

[Borodin]

Corollary. Every signed planar (simple) graph admits an edge-sign preserving homomorphism to $AM(5)$.

Proposition. With respect to switch homomorphism the core of $AM(5)$ has 40 vertices.

[Ochem, Pinlou, Sen]

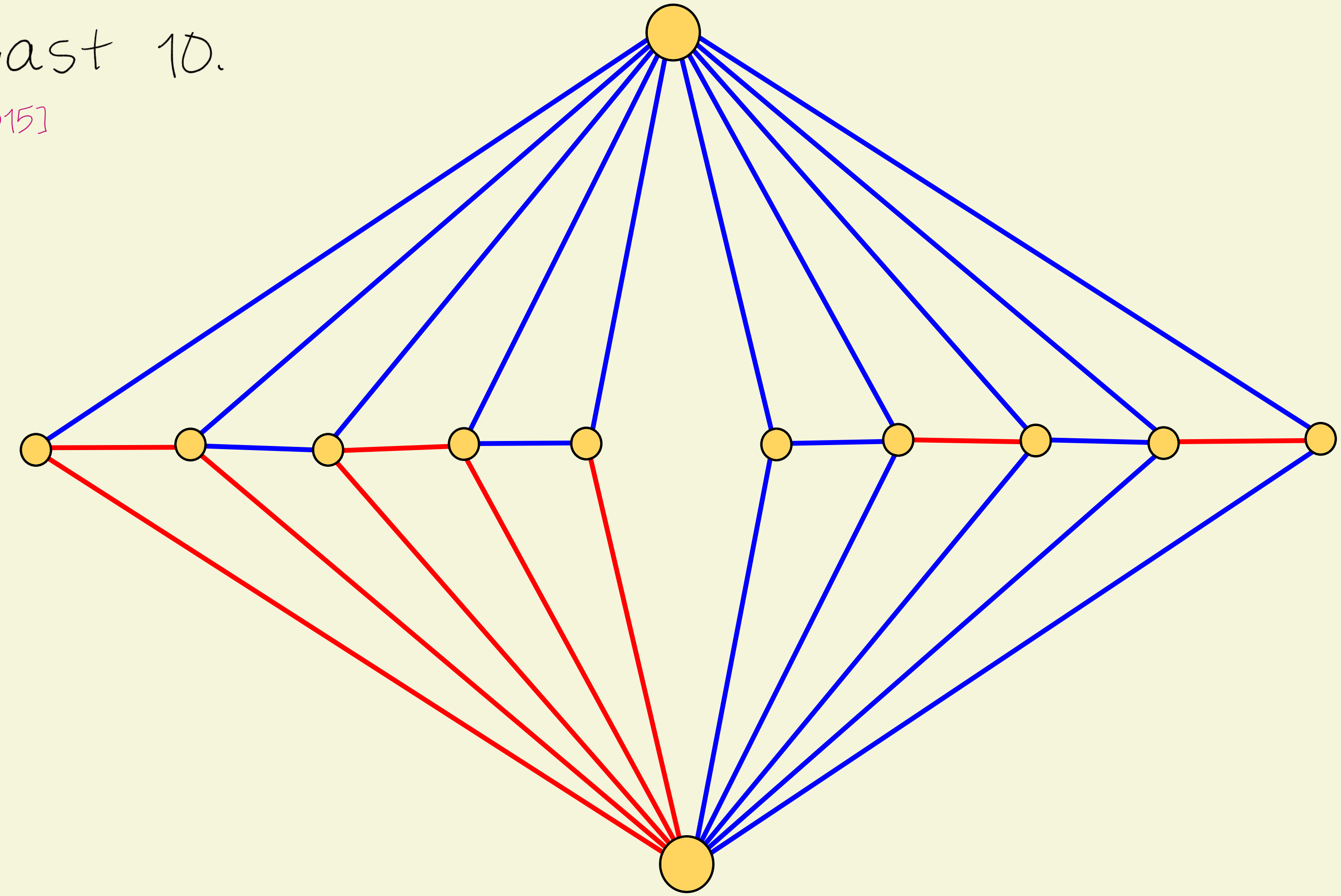
Proof is left as a homework.

Conclusion. There exists a signed simple graph on 40 vertices to which every signed planar simple graph admits a homomorphism.

Homework

Example of a signed planar simple graph where every simple homomorphic image is of order at least 10.

[Naserasr, Rollova, Sopena, 2015]



Question. Is there a signed simple graph on 10 vertices to which every signed planar simple graph admits a homomorphism?

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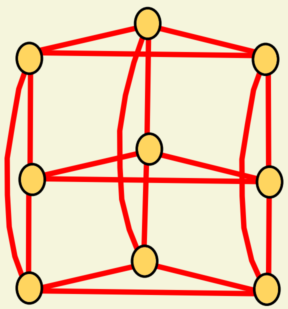
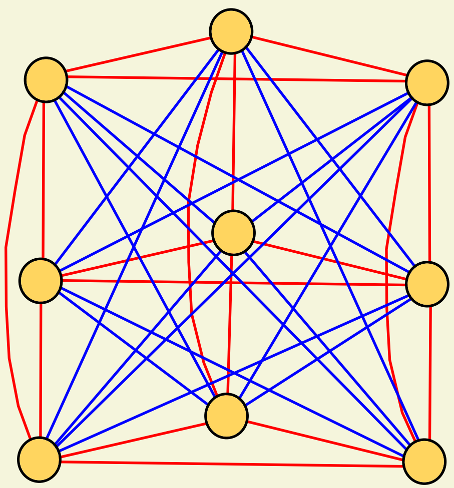
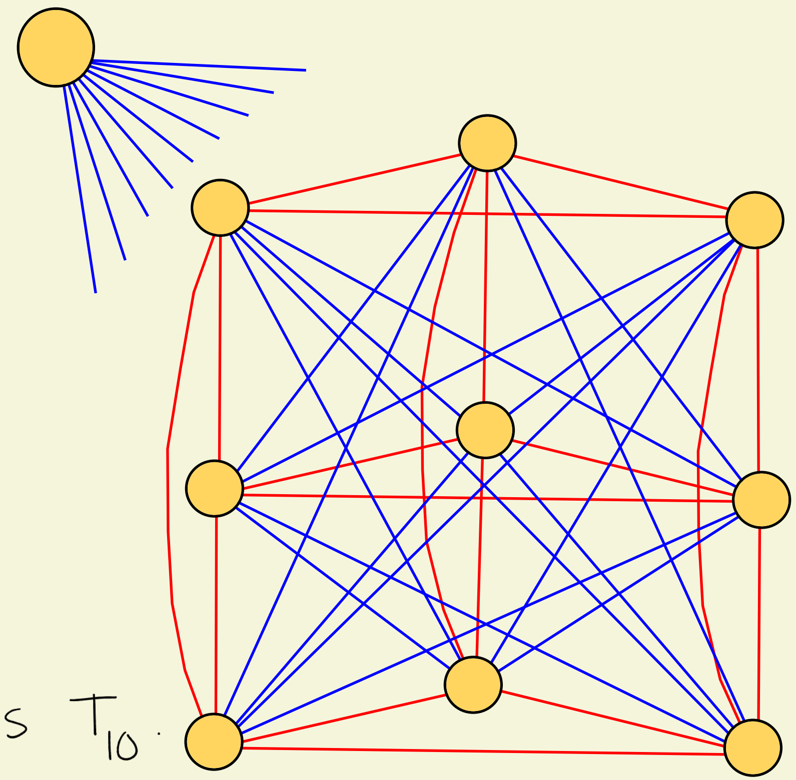


Paley GF(9)



signed Paley GF(9)

T_{10}



Theorem. The only possible 10 vertices graph is T_{10}.

[Ochem, Pinlou, Sen]

Homework. If G is a simple graph of treewidth at most 3, then for any signature σ, we have:

$$(G, \sigma) \longrightarrow T_{10}.$$